IRREDUCIBLE COMPONENTS IN AN ALGEBRAIC VARIETY OF REPRESENTATIONS OF A FAMILY OF ONE-RELATOR GROUPS

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Abstract. Given a finitely generated group G, the set $Hom(G, SL_2\mathbf{C})$ inherits the structure of an algebraic variety $R(G)$ called the *representation variety* of G. This algebraic variety is an invariant of G. Let

$$G_{pt} = \langle a, b; a^p = b^t \rangle,$$

where p, t are integers greater than one. In this paper a formula is produced yielding the number of four dimensional irreducible components of the affine algebraic variety $R(G_{pt})$. A direct consequence of the main theorem of this paper is that if K is a torus knot, then its genus equals the number of four dimensional components of the representation variety, $R(G_{pt})$, of its corresponding knot group G_{pt}.

Introduction

Let G be a finitely generated group generated by $X = \{x_1, x_2, \ldots, x_n\}$; then the set $Hom(G, SL_2\mathbf{C})$ can be endowed with the structure of an affine algebraic variety (see [BG], [LM]) here denoted by $R(G)$. The algebraic variety $R(G)$ is independent of the choice of finite set X of generators for G. In other words, if a second finite set of generators X' for G is chosen, the resulting algebraic variety is isomorphic to the initial one. It follows that $R(G)$ is an invariant of the finitely generated presentation of the group G. In the case of a free group of rank n, here denoted by F_n, the algebraic variety $R(F_n)$ is simply $(SL_2\mathbf{C})^n$, and is consequently an irreducible algebraic variety of dimension $3n$. Note that in this paper the term 'algebraic variety' may be applied to both reducible and irreducible varieties.

Let $V \subset \mathbf{C}^n$ be an algebraic variety (not necessarily reducible) and $V = V_1 \cup V_2 \cdots \cup V_n$ be its unique decomposition into maximal irreducible components (see [MD]). Then the number of maximal irreducible components of a given dimension is an invariant of V and thus, if $V = R(G)$, an invariant of the finitely generated group G.

The groups G_{pt} are known to have $Dim(R(G)) = 4$, where p, t are integers greater than one (see [LS], or the proof of Theorem A). It is the object of this communication to introduce this invariant and to produce a formula giving the number of irreducible 4-dimensional components of $R(G)$, where G is isomorphic to one of the groups

$$G_{pt} = \langle a, b; a^p = b^t \rangle.$$

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Theorem A. *Let C_4 be the number of irreducible 4 dimensional components of $R(G_{pt})$, where p and t are integers greater than one. Then*

a) $C_4 = \frac{(p-2)(t-2)+pt}{4}$ *if both p, t are even.*
b) $C_4 = \frac{(p-1)(t-1)}{2}$ *if either p or t is odd.*

That the class of groups G_{pt} in Theorem 3.1 consists of non-isomorphic groups for different pairs $p, t \geq 2$ (*up to permutation of p and t*) is a well known result of O. Schreier (1924), [SO].

Incidentally, a somewhat similar but different study, and one the author was unaware of when the main theorem of this note was obtained, is counting connected components in the space of representations of a finitely generated group in a Lie group. For example, William Goldman in [GW] develops a formula for counting the number of connected components in the space of representations of the fundamental group of an oriented surface in the n-fold covering group of $PSL(2, \mathbf{R})$ in terms of n and the genus of the surface.

Theorem A is of interest in its own right. However, one of its consequences is rather curious and deserves mention as it may suggest perhaps deeper connections between a group G and the invariants of $R(G)$. A well known fact is that a torus knot has the fundamental group which is isomorphic to some G_{pt}, where p and t are relatively prime (see [BZ]). A direct consequence of Theorem A is that if K is a torus knot then its genus equals the number of four dimensional components in $R(G_{pt})$. The genus of a torus knot can also be defined as half the rank of its commutator subgroup; consult [BZ].

Proofs

Notation. Given a positive integer p and $M \in SL_2\mathbf{C}$, denote by $\Omega(p, M)$ the set $\{A \mid A \in SL_2\mathbf{C}, A^p = M\}$. By $Tr(A)$ will be meant the trace of the matrix A. Finally, denote by I the 2×2 identity matrix.

Observation 1.

$$R(G_{pt}) = \{(m_1, m_2) | m_1 \in SL_2\mathbf{C},\, m_2 \in \Omega(t, m_1^p)\}.$$

The following proper sub-variety of $R(G_{pt})$ will play an important role in the proof of Theorem A. Let

$$S = \{(m_1, m_2) | m_1^p = \pm I\}. \tag{1.0}$$

Clearly

$$S = S_+ \cup S_-,$$

where

$$S_+ = \{(m_1, m_2) | m_1^p = I\}, \quad S_- = \{(m_1, m_2) | m_1^p = -I\}.$$

Theorem B. *$Dim((R(G_{pt}) - S)) = 3$.*

Proof. $\{R(G_{pt}) - S\}$ maps onto a quasi-affine variety Q of $R(F_1)$ via the map from $R(G_{pt})$ to $R(F_1)$ given by $\phi(m_1, m_2) = m_1$. Some care must be exercised in defining Q. If p or t is odd, it can be assumed without any loss of generality that t is the odd one. This is done since if t is even and p odd, and if $m_1^p \in Orb \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, then $\Omega(t, m_1^p) = \emptyset$, where by Orb is meant the orbit under conjugation by elements of $SL_2\mathbf{C}$; see [LS]. Note that if p is even and m in $SL_2\mathbf{C}$, then m^p is never in $Orb \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. This said, now proceed by letting

$$Q = R(F_1) - \phi(S),$$

where S is as in (1.0).

A few comments are in line regarding fibres of the map ϕ. If $m \in \phi(S)$ and $m^p = -I$, then $Dim(\phi^{-1}(m)) = 2$. This follows by Lemma 1.6 below. Similarly, if $m \in \phi(S)$ and $m^p = I$, then $Dim(\phi^{-1}(m)) = 2$ whenever $p \geq 3$. This is a consequence of Lemma 1.6 below. Any $m \in Q$ has the property that $Dim(\phi^{-1}(m)) = 0$. This is a result of the fact that if $m \in \{SL_2\mathbf{C} - \{\pm I\}\}$, and t an integer, then the equation $x^t = m$ has at most a finite number of solutions in $SL_2\mathbf{C}$; see [LS], or [GF]. As a consequence (by elementary arguments involving fibres of regular maps with a dense image between varieties (see [MD], or [LS]), since the fibres of ϕ over the quasi-affine variety Q are zero dimensional, and $(R(G_{pt}) - S)$ maps onto the quasi-affine variety Q, one obtains that

$$Dim(Q) = Dim((R(G_{pt}) - S)).$$

Notice that Q is a quasi-affine variety in the irreducible variety $SL_2\mathbf{C}$, and thus $Dim(Q) = Dim(SL_2\mathbf{C}) = 3$. This concludes the proof of Theorem B.

It follows then that if any four dimensional components are present in $R(G_{pt})$, then they are in the sub-variety S.

Next we propose to count all the four dimensional irreducible components in S, for different values of p, t. We will find necessary the following elementary lemmas stated here mostly for the convenience of the reader; their demonstrations can be obtained using naive facts from, for example, [GF] and [MD], or they can be found directly in [LS].

Lemma 1.4. *Let $A \in SL_2\mathbf{C}$ be any matrix of a given trace $b \neq \pm 2$. Then any matrix B in $SL_2\mathbf{C}$ having trace b is similar to A.*

Lemma 1.5. *Let $A \in SL_2\mathbf{C}$ be of trace $b \neq \pm 2$. Then the orbit of A under $SL_2\mathbf{C}$ conjugation is an irreducible affine algebraic variety of dimension 2.*

Lemma 1.6.

i) *If $p = 2$, then $Dim\,\Omega(p, I) = 0$, and $\Omega(p, I)$ is reducible.*
ii) *If $p > 2$, then $Dim\,\Omega(p, I) = 2$, and $\Omega(p, I)$ is reducible.*
iii) *If $p \geq 2$, $Dim\,\Omega(p, -I) = 2$, and for for $p > 2$, $\Omega(p, -I)$ is reducible.*

From Observation 1 together with (1.0), it is easy to deduce that

$$S_+ = \Omega(p, I) \times \Omega(t, I). \tag{1.1}$$

$$S_- = \Omega(p, -I) \times \Omega(t, -I). \tag{1.2}$$

Using Lemmas 1.4 and 1.5 it is possible to deduce that the number of two dimensional irreducible components in $\Omega(p, I)$ is given by:

$$\frac{p-1}{2}, \text{ if } p \text{ is odd,} \tag{1.3}$$

$$\frac{p-2}{2}, \text{ if } p \text{ is even.} \tag{1.4}$$

In a similar fashion one can be deduced that the number of two dimensional irreducible components in $\Omega(p, -I)$ is given by:

$$\frac{p-1}{2}, \text{ if } p \text{ is odd,} \tag{1.5}$$

$$\frac{p}{2}, \text{ if } p \text{ is even.} \tag{1.6}$$

Note, having made these deductions, it is clear that the $R(G_{pt})$ are four dimensional reducible varieties.

Proof of Theorem A. If both p and t are even then using (1.1) and (1.2) together with (1.4) and (1.6) the number of four dimensional components in $R(G_{pt})$ is given by:

$$(\frac{p-2}{2})(\frac{t-2}{2}) + (\frac{p}{2})(\frac{t}{2}).$$

If both p and t are odd then the number of four dimensional components is given by:

$$(\frac{p-1}{2})(\frac{t-1}{2}) + (\frac{p-1}{2})(\frac{t-1}{2}).$$

In the case that p is odd and t is even then the number of four dimensional components is given by:

$$(\frac{p-1}{2})(\frac{t-2}{2}) + (\frac{p-1}{2})(\frac{t}{2}).$$

The proof of Theorem A is now complete since elementary algebra yields the quantities promised in the statement of Theorem A.

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